RESEARCH REPORT

Automorphisms of Baer-Levi Semigroups

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No. 19

November 1981

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New Zealand

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AUTOMORPHISMS OF BAER-LEVI SEMIGROUPS

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Let T_X be the semigroup, under composition, of all transformations of the set X to itself, and G_X be the group, inside T_X, of all bijections of X. An automorphism ϕ of a semigroup S in T_X is said to have the *inner automorphism property* (i.a.p.) if the automorphisms of S are precisely those of the form $\phi(f) = hfh^{-1}$, for all f in S, where h is an element of G_X. There is a readily stated unsolved problem concerning T_X: determine all subsemigroups S of T_X which have the inner automorphism property.

Amidst research on this problem there are two landmarks, the works of Schreier [5] and Fitzpatrick and Symons [2]. Schreier showed that if S contains the semigroup of all constant maps (I_X), then S has the i.a.p., while Fitzpatrick and Symons showed that for semigroups S containing G_X the i.a.p. holds. A large family of semigroups (first considered in [1]) which are disjoint from I_X and G_X are the Baer-Levi semigroups. In this paper we show that such semigroups also possess the i.a.p.

The family of Baer-Levi semigroups are defined in the following way. Let $|X|$, the cardinality of the set X, be p, and q be an infinite cardinal less than or equal to p. Then

$$BL(p,q) = \{f \in T_X: \ f \text{ is one-to-one and } |X \setminus R(f)| = q\}$$

is the Baer-Levi semigroup of type (p,q). All congruences on $BL(p,p)$ were found by Sutov [6], while those on $BL(p,q)$ were found by Lindsey and Madison [3].

Let $B(p,q) = \{A \subseteq X: \ |A| = p \text{ and } |X \setminus A| = q\}$. We call $B(p,q)$ the family of all *Baer-Levi sets of type* (p,q). We view $B(p,q)$ as a partially ordered set, the order being given by set inclusion. A central part of our proof is a result of independent interest concerning $B(p,q)$, so we present it in the form of a lemma. For this we need a definition: a bijection H of $B(p,q)$ is said to be *induced* by the bijection h of X if $H(A) = h(A) (=\{h(x): x \in A\})$ for every A in $B(p,q)$.

LEMMA: Let H be a bijection of $B(p,q)$. Then H is induced if and only if H and H^{-1} are order-preserving.

PROOF: If H is induced it is clear that H and H^{-1} are order-preserving. We show the converse in four steps.

1. *Let* $A, B \in \mathcal{B}(p,q)$ *with* $A \subseteq B$ *and* $|B \backslash A|$ *finite.*
Then $|B\backslash A| = |H(B)\backslash H(A)|$.

If $|B\backslash A| = n$, then $|\{S: A \subseteq S \subseteq B\}| = 2^n$.
But H and H^{-1} are order preserving bijections so $|\{H(S): H(A) \subseteq H(S) \subseteq H(B)\}| = 2^n$ also. Thus $|\{T: H(A) \subseteq T \subseteq H(B)\}| = 2^n$ and hence $|H(B)\backslash H(A)| = n$.

2. *Given* $x \in X$ *there exists a* $y \in X$ *such that*
$H(B_\cup\{x\}) = H(B)_\cup\{y\}$ *for every* $B \in \mathcal{B}(p,q)$ *with* $x \notin B$.

For brevity we write $B_\cup\{x\}$ *as* $B_\cup x$ in future. Take $A \in \mathcal{B}(p,q)$ with the given x not in A. Step 1 and the fact that H is order preserving together imply

$$H(A_\cup x) = H(A)_\cup y \text{ for some } y \text{ in } X.$$

We show $H(B_\cup x) = H(B)_\cup y$ for every other $B \in \mathcal{B}$ with $x \notin B$. This is done in three stages.

i) *Case* $B \subseteq A$: Let $H(B_\cup x) = H(B)_\cup z$. Now $B_\cup x \subseteq A_\cup x$ so $H(B_\cup x) \subseteq H(A_\cup x)$ or $H(B)_\cup z \subseteq H(A)_\cup y$ so $Z \in H(A)_\cup y$. But $Z \notin H(A)$ (for then $H(B_\cup x) \subseteq H(A)$ whence $B_\cup x \subseteq A$, since H^{-1} is order preserving, a contradiction), so $z = y$.

ii) *Case* $A_\cap B \in \mathcal{B}(p,q)$: For this case we need the following small result: if $H(C_\cup x) = H(C)_\cup y$, then $H(C_\cup D_\cup x) = X(C_\cup D)_\cup y$, where $C \in \mathcal{B}(p,q)$, $C_\cup D \in \mathcal{B}(p,q)$, $C_\cap D = \phi$ and $x \notin C_\cup D$. The proof is as follows.

Suppose $y \in H(C_\cup D)$. Certainly $H(C) \subseteq H(C_\cup D)$ so $H(C_\cup x) \subseteq H(C_\cup D)$, hence $C_\cup x \subseteq C_\cup D$, a contradiction, so $y \notin H(C_\cup D)$. Now let $H(C_\cup D_\cup x) = H(C_\cup D)_\cup z$. We have $C_\cup x \subseteq C_\cup D_\cup x$ so $H(C_\cup x) \subseteq H(C_\cup D_\cup x)$ or $y \in H(C_\cup D_\cup x) = H(C_\cup D)_\cup z$. Since $y \notin H(C_\cup D)$, $y = z$.

Returning to the case $A_\cap B \in \mathcal{B}(p,q)$, we know $H((A_\cap B)_\cup x) = H(A_\cap B)_\cup y$ from i). Letting $C = A_\cap B$ and $D = B \backslash A$ in the small result now gives $H(B_\cup x) = H(B)_\cup y$ as required.

iii) *Case* $A_\cap B \notin \mathcal{B}(p,q)$: Suppose $q < p$. Then either $|A_\cap B| = p$ and $|(A_\cap B)'| \neq q$, whence $|A_\cap B'| = |A'_\cup B'| \neq q$, contradicting $|A'| = |B'| = q$, *or* $|A \cap B| < p$, whence $|(A_\cap B)'| = |A'_\cup B'| = p$, so one of $|A'|, |B'|$ is p, again a contradiction. Thus in this case we have $q = p$. Then we can find a $C \in \mathcal{B}(p,p)$ with $x \notin C$ such that $A_\cap C$ and $B_\cap C \in \mathcal{B}(p,p)$. Then $H(A_\cup x) = H(A)_\cup y$ implies $H(C_\cup x) = H(C)_\cup y$, using ii) and $A_\cap C \in \mathcal{B}(p,p)$. But in turn this implies $H(B_\cup x) = H(B)_\cup y$, using ii) and $B_\cap C \in \mathcal{B}(p,p)$.

We are now able to produce the required bijection of X:

Definition: Given $x \in X$, define a mapping $h: X \to X$ by $h(x) = y$, where $H(B_\cup x) = H(B)_\cup y$ for some $B \in \mathcal{B}(p,q)$ with $x \notin B$.

3. *h is a well-defined bijection of X.*

Step 2 ensures h is well-defined. Now suppose $h(x) = h(x') = y$, say and take $B \in \mathcal{B}(p,q)$ with $x, x' \notin B$. Then

$$H((B \cup x) \cup x') = H(B \cup x) \cup y = H(B) \cup y = H(B \cup x).$$

Since H is one-to-one we must have $x = x'$, so h also is one-to-one.

Finally, take $y \in X$ and $C \in \mathcal{B}(p,q)$ with $y \notin C$. Consider $H^{-1}(C \cup y) = H^{-1}(C) \cup x$, for some x. Then $H(H^{-1}(C) \cup x) = C \cup y$ so $h(x) = y$, or h is onto.

4. *H is induced by h.*

We must show $H(A) = h(A)$, for each $A \in \mathcal{B}(p,q)$, where $h(A) = \{h(x) : x \in A\}$. From the definition of h we at once have $h(A) \subseteq H(A)$. Take $y \in H(A)$. Then $H^{-1}(H(A) \setminus y) = A \setminus x$, for some $x \in A$, so $H(A \setminus x) = H(A) \setminus y$ or $h(x) = y$. Thus $H(A) \subseteq h(A)$, so equality follows.

<u>THEOREM</u>: $BL(p,q)$ has the inner automorphism property.

<u>PROOF</u>: $BL(p,q)$ is certainly G_X-normal, so it suffices to show that every automorphism ϕ of $BL(p,q)$ has the form $\phi(f) = hfh^{-1}$, for all $f \in BL(p,q)$, and some fixed bijection h of X. This is carried out in four steps. Throughout, f, g and k are elements of $BL(p,q)$.

1. *$R(f) \subseteq R(g)$ if and only if for each k such that $kg = g$ we have $kf = f$.*

Suppose $R(f) \subseteq R(g)$. Then $kg = g$ implies k is the identity on $R(g)$, so also on $R(f)$. Hence $kf = f$.

Suppose now $R(f) \not\subseteq R(g)$ $(= A$ say$)$. Let $\{B_1, B_2, \ldots\}$ be a partition of A' such that $|B_i| = q$, and $k_i \colon B_i \to B_{i+1}$ an arbitrary bijection, for each $i \geqslant 1$. Then k, given by $k(x) = x$ for $x \in A$ and $k(x) = k_i(x)$ for $x \in B_i$, each i, lies in $BL(p,q)$ and has fixed points precisely $R(g)$. Thus $kg = g$, yet $kf \neq f$.

2. *$R(f) = R(g)$ if and only if $R(\phi(f)) = R(\phi(g))$*

Using the result of step 1 we immediately have that $R(f) \subseteq R(g)$ if and only if $R(\phi(f)) \subseteq R(\phi(g))$, from which step 2 follows.

Thus the automorphism ϕ gives rise in a natural way to a mapping of $BL(p,q)$:

Definition: Given $A \in \mathcal{B}(p,q)$, define $H(A) = R(\phi(f))$, where f in $BL(p,q)$ is such that $R(f) = A$.

3. *H is a well-defined bijection of $\mathcal{B}(p,q)$, with H and H^{-1} order-preserving.*

That H is well-defined is the content of step 2. Suppose $A \neq B$, $A, B \in \mathcal{B}(p,q)$. Then if $R(f) = A$ and $R(g) = B$ we have $R(\phi(f)) \neq R(\phi(g))$, by step 2, so $H(A) \neq H(B)$, or H is one-to-one. Now take $B \in \mathcal{B}(p,q)$

and f such that $R(\phi(f)) = B$. If $A = R(f)$ we must have $H(A) = R(\phi(f)) = B$, so H is onto.

Finally, the definition of H, together with the fact that $R(f) \subseteq R(g)$ if and only if $R(\phi(f) \subseteq R(\phi(g))$, ensures that H and H^{-1} are order-preserving.

4. ϕ *is inner.*

From the lemma we now have that H is induced by a bijection h of X. We show that $\phi(f) = hfh^{-1}$ for each f in $BL(p,q)$.

Take such an f and an $x \in X$ and suppose $f(x) = y$. Choose A and B in $B(p,q)$ such that $A \subseteq B$ and $B \setminus A = \{x\}$, together with p and q in $BL(p,q)$ such that $R(p) = A$ and $R(q) = B$.

Now $R(q) \setminus R(p) = B \setminus A = \{x\}$ so $R(\phi(q)) \setminus R(\phi(p)) = H(B) \setminus H(A) = \{h(x)\}$. On the other hand, $R(fq) \setminus R(fp) = \{y\}$ so $R(\phi(fq)) \setminus R(\phi(fp)) = \{h(y)\}$. But since $R(\phi(fq)) \setminus R(\phi(fp)) = R(\phi(f)\phi(q)) \setminus R(\phi(f)\phi(p))$ we must have $\phi(f)h(x) = h(y) = hf(x)$. Thus $\phi(f) = hfh^{-1}$.

On completion of this work the authors discovered that the result has been announced in [4]. Schein's quite different proof, yet to appear, involves showing that ϕ permutes the subsemigroups

$$S_x = \{f \in BL(p,q) : f(x) = x\},$$

where $x \in X$.

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